JOURNAL OF ECONOMIC THEORY 40, 319–328 (1986)

Notes, Comments, and Letters to the Editor

On the Existence of a Stationary Optimal Stock for a Multi-sector Economy: A Primal Approach*

M. ALI KHAN

Department of Economics, University of Illinois at Urbana-Champaign, Champaign, Illinois 61820

AND

TAPAN MITRA

Department of Economics, Cornell University, Ithaca, New York 14853

Received March 19, 1984; revised November 15, 1985

Using a purely primal approach, we show the existence of a non-trivial stationary optimal stock for a multi-sector economy. Our result generalizes earlier work by dropping the δ-productivity assumption and by replacing the continuity hypothesis on the utility function by upper semicontinuity. *Journal of Economic Literature* Classification Number: 111. © 1986 Academic Press, Inc.

1. INTRODUCTION

The concept of a non-trivial stationary optimal stock (SOS) plays a central role in the theory of optimal intertemporal allocation and its existence in a multi-sector model has been shown by Sutherland [14], Hansen and Koopmans [6], Peleg and Ryder [12], Cass and Shell [2], McKenzie [8, 9], Flynn [5], among others.

The demonstration of existence typically consists of three separate steps. First, a fixed point argument is used to show the existence of what we call in the sequel, a discounted golden-rule stock. Second, a separation

* In preparing this manuscript, we have benefitted from the comments of Lionel McKenzie and Yaw Nyarko and from conversations with Prajit Dutta, Mukul Majumdar, and Makoto Yano. The research of the second author was supported by a National Science Foundation Grant and an Alfred P. Sloan Research Fellowship.

0022-0531/86 $3.00
Copyright © 1986 by Academic Press, Inc.
All rights of reproduction in any form reserved.

argument in the form of the Kuhn–Tucker theorem is used to provide a "price-support" to the discounted golden-rule stock. Finally, a computation based on the price support property is used to show that the discounted golden-rule stock is optimal among all programs starting from that stock.

This is the approach of Cass and Shell [2], McKenzie [8, 9] and Flynn [5]. Peleg and Ryder [12] also rely on the Kakutani and Kuhn–Tucker theorems though they combine the first two steps and differ in terms of the details of the third. An exception to this is the work of Sutherland [14] who relies on methods of dynamic programming and is able to avoid supporting prices and the Kuhn–Tucker theorem. However, Sutherland does not establish the existence of a *non-trivial* SOS, and as noted by Peleg and Ryder [12], the null stock is always a SOS in a set-up which allows for the possibility of inaction and the impossibility of getting positive outputs from zero inputs.

The prevalence of duality methods for the existence results is rather striking. In proposing a "primal" approach to "turnpike theory," McKenzie [9] remarks, "The use of duality in establishing the existence of an optimal stationary path seems harder to avoid than in proving asymptotic theorems."

In this note, we propose a purely primal approach to the existence of a non-trivial SOS. Our proof avoids the Kuhn–Tucker theorem and by a simple computation based on Jensen's inequality, we can directly establish that a discounted golden-rule stock is a non-trivial SOS. The application of Jensen's inequality is, of course, not new and is implicit in Brock [1] and more recently, explicitly used in Dechert and Nishimura [4]. However, its relevance to the existence problem studied here seems to have been overlooked.

A direct payoff of our approach is that in dispensing with the Kuhn–Tucker theorem we no longer need Slater's constraint qualification in the form of the δ-productivity assumption, i.e., the existence of a feasible input–output pair (x, y) such that δy is greater in *all* coordinates than x, where δ is the discount factor. In the context of Flynn's work [5], our result shows that the δ-productivity assumption can simply be dropped from his theorem. The comparison with McKenzie's work is less clear since he does not show the existence of a non-trivial SOS (in our sense) for his model (see Example 3 and Remark 2 below). Moreover, in assuming that the technology set is not necessarily closed, he makes use of the δ-productivity assumption even for the fixed point argument. We do, however, present an example in which δ-productivity is violated but a non-trivial SOS exists (see Example 2).

Our result also generalizes the Flynn–McKenzie theorem by replacing the continuity hypothesis on the utility function by upper semicontinuity. Such an extension is motivated by a class of economies considered by Peleg in which the utility functions are not continuous but only upper semicontinuous; see Peleg [10, Remark 2].

A secondary contribution of this note is to use a purely primal approach to show that a non-trivial SOS, k, is a discounted golden-rule stock, provided (k, k) is in the interior of the technology set. This result is proved by McKenzie [8] relying on duality methods. Again, the proof involves three steps. First, a sequence of prices is found to support the stationary optimal program, following the approach of Weitzman [15], or Peleg and Ryder [11]. Second, by an argument due to Sutherland [13], a "quasi-stationary" price support (i.e., $p(t) = \delta^t p$ for $t \geqslant 0$) is obtained from the above sequence of supporting prices. Third, this (quasi-stationary) price support property is used to show that the SOS is a discounted golden-rule stock. In dispensing with support prices, we provide a direct and short proof. We also present an example to show that the result fails when (k, k) is not in the interior of the technology set (see Example 1).

2. Preliminaries

2a. *Notation*

We shall be working in n-dimensional Euclidean space R^n, where $\|x\|$ denotes the Euclidean norm of any element x in R^n. For any x, y in R^n, we shall write $x \gg y$ $(x \geqslant y)$ to denote $x_i > y_i$ $(x_i \geqslant y_i)$ for all coordinates $i = 1,..., n$; and $x > y$ to denote $x \geqslant y$ and $x \neq y$. Let R^n_+ be the non-negative orthant of R^n, i.e., $R^n_+ = \{x \in R^n: x \geqslant 0\}$. For any set, S, $\mathfrak{P}$ (S) denotes the set of all subsets of S and hence we shall write $\phi: X \to \mathfrak{P}(Y)$ for any correspondence (set-valued map) ϕ with domain X and range $\mathfrak{P}(Y)$. Finally, let e denote an element of R^n_+, all of whose coordinates are unity.

2b. *The Model*

An *economy* $\mathfrak{E}$ consists of a triple $(\mathfrak{T}, u, \delta)$, where $\mathfrak{T} \in \mathfrak{P}$ $(R^n_+ \times R^n_+)$ is the technology, u is a utility function with domain $\mathfrak{T}$ and range R, and δ a discount factor such that $0 < \delta < 1$. We shall need the following assumptions on $\mathfrak{E}$:

A1. (i) $(0, 0) \in \mathfrak{T}$; (ii) $(0, y) \in \mathfrak{T}$ implies $y = 0$.

A2. $\mathfrak{T}$ is (i) closed and (ii) convex.

A3. There is β such that $\|x\| > \beta$ for any $(x, y) \in \mathfrak{T}$ implies $\|y\| \leqslant \|x\|$.

A4. $(x, y) \in \mathfrak{T}$ implies $(z, w) \in \mathfrak{T}$ for all $z \geqslant x$ and $0 \leqslant w \leqslant y$. Moreover, $u(z, w) \geqslant u(x, y)$.

A5. u is (i) upper semicontinuous $[(x^n, y^n) \to (x, y)$ imply $\limsup_{n \to \infty} u(x^n, y^n) \leqslant u(x, y)]$ and (ii) concave.

Except for A5 (i), these assumptions are all standard. It may be worth pointing out, however, that McKenzie [9] does not assume A1, the upper semicontinuity of u and the closedness of $\mathfrak{T}$. Instead, he assumes a closedness condition on u and a boundedness assumption on $\mathfrak{T}$; see I and II in McKenzie [9, p. 196].

We now state the following basic concepts for our economy $\mathfrak{E}$:

D.1 A *program* starting from an initial vector $\bar{k} \in R^n_+$ is a sequence $\{k(t)\}_0^\infty$ such that $k(0) = \bar{k}$ and $(k(t), k(t+1)) \in \mathfrak{T}$ for all $t = 0, 1, 2, \ldots$.

D.2 A program $\{k(t)\}_0^\infty$, starting from $\bar{k} \in R^n_+$ is said to be an *optimal program* if for any other program $\{k'(t)\}_0^\infty$ starting from $\bar{k}$, we have $\sum_{t=0}^\infty \delta^t u(k(t), k(t+1)) \geqslant \sum_{t=0}^\infty \delta^t u(k'(t), k'(t+1))$.

D.3 An optimal program $\{k(t)\}_{t=0}^\infty$ starting from $\bar{k} \in R^n_+$ is said to be a *stationary optimal program* if $k(t) = \bar{k}$ for all t.

D.4 A *stationary optimal stock* k is an element of R^n_+ such that $\{k\}_0^\infty$ is a stationary optimal program. It is said to be *non-trivial* if $u(k, k) > u(0, 0)$.

D.5 A *discounted golden-rule stock* k is an element' of R^n_+ such that

(i) $(k, k) \in \mathfrak{T}$

(ii) $u(k, k) \geqslant u(x, y)$ for all $(x, y) \in \mathfrak{T}$ such that $x \leqslant (1-\delta) k + \delta y$

(iii) $u(k, k) > u(0, 0)$.

2c. *Existence of Optimal Programs and the Principle of Optimality*

Our first result is on the existence of an optimal program. The proof, being fairly standard, is omitted.

THEOREM 1. *Under* A1, A3, *and* A4, *for any program* $\{k(t)\}_0^\infty$ *starting from* $\bar{k}$, *we have* $\|k(t)\| \leqslant \max[\beta, \|\bar{k}\|] \equiv B(\bar{k})$ *for all* t, *where* β *is taken from* A3. *Under the additional assumption* A5 (i), *there exists an optimal program starting from any given initial vector* $\bar{k}$.

Under the assumptions of Theorem 1, there is an optimal program $\{k^*(t)\}_0^\infty$ from each $k \in R^n_+$. We define $V(k) = \sum_{t=0}^\infty \delta^t u(k^*(t), k^*(t+1))$; V is generally known as the value function. The following result is standard and is known as the "principle of optimality."

LEMMA 1. *If* $\{k(t)\}_0^\infty$ *is an optimal program from* k, *then*

$$V(k) = \sum_{t=0}^{N} \delta^t u(k(t), k(t+1)) + \delta^{N+1} V(k(N+1)) \textit{ for } N \geqslant 0.$$

3. Equivalence of Discounted Golden-Rule and Non-trivial Stationary Optimal Stocks

The equivalence of a discounted golden-rule stock and a non-trivial SOS is given in McKenzie [8, p. 42]. Our treatment is primal in that it makes no use of supporting prices.

Theorem 2. *Under* A1–A5, *every discounted golden-rule stock k is a non-trivial stationary optimal stock.*

Proof. The fact that $u(k, k) > u(0, 0)$ is true by hypothesis. Now let $\{k(t)\}_0^\infty$ be any program starting from k. We shall show that it does not give a higher utility than the path $\{k\}_0^\infty$.

Let $x(T) = \sum_{t=0}^{T-1} (1-\delta)\ \delta^t k(t)/(1-\delta^T)$ and $y(T) = \sum_{t=0}^{T-1} (1-\delta)\ \delta^t k(t+1)/(1-\delta^T)$. Given convexity of $\mathfrak{T}$, certainly $(x(T), y(T)) \in \mathfrak{T}$ for all $T \geq 1$. From Theorem 1, we know that $k(t)$ is bounded independently of t. Hence $(\bar{x}, \bar{y}) = \operatorname{Lim}_{T\to\infty} (x(T), y(T))$ is well defined and by virtue of A2(i), is an element of $\mathfrak{T}$.

Now, by A5 and the fact that $0 < \delta < 1$, Jensen's inequality yields $u(\bar{x}, \bar{y}) \geq \sum_{t=0}^\infty (1-\delta)\ \delta^t u(k(t), k(t+1))$. But $(\bar{x} - \delta\bar{y}) = (1-\delta)[\sum_{t=0}^\infty \delta^t k(t) - \sum_{t=0}^\infty \delta^{t+1} k(t+1)] = (1-\delta)\, k$. Since (k, k) is a discounted golden-rule stock, certainly $u(k, k) \geq u(\bar{x}, \bar{y})$, which implies $\sum_{t=0}^\infty \delta^t u(k, k) \geq \sum_{t=0}^\infty \delta^t u(\bar{x}, \bar{y}) = u(\bar{x}, \bar{y})/(1-\delta) \geq \sum_{t=0}^\infty \delta^t u(k(t), k(t+1))$.

We can now state a converse to Theorem 2.

Theorem 3. *Under* A1(i), A2(ii), A4, *and* A5(ii), *every non-trivial stationary optimal stock k such that $(k, k) \in$ interior $\mathfrak{T}$, is a discounted golden-rule stock.*

Proof. Suppose not; then there exists $(x, y) \in \mathfrak{T}$ such that $x \leq (1-\delta)\, k + \delta y$ and $u(x, y) > u(k, k)$. Since u is non-decreasing in the first component by virtue of A4, we can assume without any loss of generality that $x = (1-\delta)\, k + \delta y$. Let $\gamma = u(x, y) - u(k, k) > 0$.

Using (x, y), we shall now construct a program $\{k(t)\}_0^\infty$ starting from k that gives more utility than the stationary optimal program $\{k\}_0^\infty$. This furnishes us the required contradiction. Towards this end, for a value of N to be determined later, let

$$(z(q), z(q-1)) = (1-\delta^q)(k, k) + \delta^q (x, y), \qquad q = 1, \ldots, N. \tag{1}$$

By A2 (ii), $(z(q), z(q-1)) \in \mathfrak{T}$ for all $q = 1, \ldots, N$. Now let $\{k(t)\}_0^\infty$ be such that $k(0) = k$; $k(t) = z(N-t+1)$, $t = 1, \ldots, N$; $k(N+1) = z(0) = x$; $k(t) = 0$, $t \geq N+2$.

We can show that for large enough N, $\{k(t)\}_0^\infty$ is a program (in the sense of D.1). For this, it only remains to show that $(k, k(1)) = (k, z(N)) \in \mathfrak{T}$. But $(k, k) \in \text{interior}\, \mathfrak{T}$ and so there exists $\alpha > 0$ such that $(k, y) \in \mathfrak{T}$ for all $y \in S_2 \equiv \{y: k - 2\alpha e \leqslant y \leqslant k + 2\alpha e\}$. Let $S_1 = \{y: k - \alpha e \leqslant y \leqslant k + \alpha e\}$. On substituting the value of x in (1), it is clear that

$$z(q) - \delta z(q-1) = (1-\delta)\, k \qquad \text{which implies } (z(q) - k) = \delta(z(q-1) - k).$$

Since δ is less than 1, certainly $z(q) \to k$ as $q \to \infty$ and hence there exists N_1 such that $z(N) \in S_1$ for all $N \geqslant N_1$.

Next, we can assert, as a consequence of A5 (ii) that, for all $q = 1,...N$,

$$u(z(q), z(q-1)) \geqslant (1 - \delta^q)\, u(k, k) + \delta^q u(x, y) \geqslant u(k, k) + \delta^q \gamma.$$

By Mangasarian (7, p. 63), it is also true that

$$\|u(k, z(N)) - u(k, k)\| \leqslant A \|z(N) - k\| = A\delta^{N+1} \|y - k\|,$$

where $A \equiv (u(k, k) + \hat{\beta})/\alpha$, $\hat{\beta} = -\text{Min}_{y \in W}\, u(k, y)$ and W is the set of $2n$ vertices of S_1. Hence we have

$$\sum_{t=0}^{N+1} \delta^t [u(k(t), k(t+1)) - u(k, k)] \geqslant -A\delta^{N+1} \|y - k\| + (N+1)\, \delta^{N+1} \gamma.$$

On adding terms after the time period $(N+1)$, we obtain, with V as in Lemma 1,

$$\sum_{t=0}^{\infty} \delta^t [u(k(t), k(t+1)) - u(k, k)]$$
$$\geqslant \delta^{N+1}((N+1)\,\gamma - A\|y-k\| + \{\delta u(0,0)/(1-\delta)\} - \delta V(k)). \quad (2)$$

Let N_2 be a value of N such that the right-hand side of (2) is positive. Let $N' = \text{Max}(N_1, N_2)$. Now any $\{k(t)\}_0^\infty$ with $N \geqslant N'$ furnishes us with a contradiction to the fact that $\{k\}_0^\infty$ is a stationary optimal program. Since k is a non-trivial stationary program, $u(k, k) > u(0, 0)$ and the proof is finished.

Remark 1. The proof is valid if instead of the convexity of $\mathfrak{T}$ and the concavity of u over $\mathfrak{T}$, we assume only that $\mathfrak{T}_l = \{(x, y): x = (1-\delta)\, k + \delta y\}$ is convex and that u is concave over $\mathfrak{T}_l$.

A natural question arises as to whether the interiority hypothesis in Theorem 3 can be dispensed with. The following example shows this not to be the case.

EXAMPLE 1. Let $\mathfrak{T} = \{(x, y) \in R_+^2 \times R_+^2: \quad Ay \leqslant x, \quad ey \leqslant 3\}$, where

$A = \begin{pmatrix} 1 & 0 \\ 1 & 1/2 \end{pmatrix}$. Let $\delta = \frac{1}{2}$ and $u(x, y) = ex$. It is clear that this economy satisfies A1–A5. We shall show that $k = (1, 0)$ is a non-trivial stationary optimal stock. It is non-trivial because $u(k, k) = ek = 1 > u(0, 0) = 0$. To show that it is stationary optimal stock, observe that $(k, k) \in \mathfrak{T}$ and consider any program $\{k(t)\}_0^\infty$ starting from k. Since $(k(t), k(t+1)) \in \mathfrak{T}$, $k(t) \leqslant (1, 0)$ for all t. Hence

$$\sum_{t=0}^{\infty} \delta^t u(k(t), k(t+1)) = \sum_{t=0}^{\infty} \delta^t (ek(t)) \leqslant \sum_{t=0}^{\infty} \delta^t$$
$$= \sum_{t=0}^{\infty} \delta^t (ek) = \sum_{t=0}^{\infty} \delta^t u(k, k).$$

Now let $x' = (1, 1)$, $y' = (1, 2)$. Certainly $(x', y') \in \mathfrak{T}$ and $\delta y' - x' = (\delta - 1) k$. But $u(x', y') = ex' = 2 > ek = u(k, k)$ and thus k is not a discounted golden-rule stock.

4. Existence of Discounted Golden-Rule and Non-trivial Stationary Optimal Stocks

We now turn to the existence issue with Theorem 2 in McKenzie [9, p. 199] as the relevant benchmark. We shall need the following definition for our next result.

D.6 An economy is *δ-normal* if there exists $(\bar{x}, \bar{y}) \in \mathfrak{T}$ such that $\bar{x} \leqslant \delta \bar{y}$ and $u(\bar{x}, \bar{y}) > u(0, 0)$.

THEOREM 4. *If $\mathfrak{E}$ satisfies* A1–A5 *and is δ-normal, there exists a discounted golden-rule stock.*

The proof of Theorem 4 relies heavily on the following result.

LEMMA 2. *Let $S = \{x \in R_+^n : \|x\| \leqslant \beta\}$ and ϕ and ψ be mappings from S into $\mathfrak{P}(R_+^n \times R_+^n)$ such that for $z \in S$, $\phi(z) = \{(x, y) \in \mathfrak{T}: x \leqslant (1 - \delta) z + \delta y\}$ and $\psi(z) = \{(x, y) \in \phi(z): u(x, y) \geqslant u(x', y')$ for all $(x', y') \in \phi(z)\}$. If $\mathfrak{E}$ satisfies* A1–A5, *ψ is a non-empty, convex-valued, and upper semicontinuous correspondence.*

Proof. Clearly, S is a non-empty, convex, and compact set. Next, we claim that ϕ is a non-empty, convex and compact-valued correspondence. For any $z \in S$, $(0, 0) \in \phi(z)$, and, since $\mathfrak{T}$ is convex and closed, $\phi(z)$ is convex and closed. Furthermore, if $(x, y) \in \phi(z)$, then $\|x\| \leqslant \beta$. (Otherwise, if $\|x\| > \beta$, then by A3, $\|x\| \leqslant (1 - \delta)\|z\| + \delta\|y\| \leqslant (1 - \delta)\|z\| + \delta\|x\|$, so $\|x\| \leqslant \|z\| \leqslant \beta$, a contradiction.) This implies by Theorem 1 that if $(x, y) \in \phi(z)$, then $\|y\| \leqslant \beta$. Thus on defining $S' = \{(x, y) \in R^n_+ \times R^n_+ : \|x\| \leqslant \beta, \|y\| \leqslant \beta\}$, we note that S' is a non-empty, compact set, and for any $z \in S$, $\phi(z)$ is a subset of S'. Since $\phi(z)$ is closed for each $z \in S$, $\phi(z)$ is compact for each $z \in S$.

Since u is an upper semicontinuous function on $\mathfrak{T}$, and $\phi(z)$ is a non-empty, compact subset of $\mathfrak{T}$, $\psi(z)$ is non-empty for each $z \in S$. It is also convex as a consequence of A5 (ii) and of the convexity of $\phi(z)$.

Next, we show the upper semicontinuity of ψ. Let z^* be an arbitrary point of S. Consider a sequence $z^n \in S$, with $z^n \to z^*$ as $n \to \infty$. Let $(x^n, y^n) \in \psi(z^n)$, and $(x^n, y^n) \to (\hat{x}, \hat{y})$. We want to show that $(\hat{x}, \hat{y}) \in \psi(z^*)$. Since $\mathfrak{T}$ is closed, $(\hat{x}, \hat{y}) \in \phi(z^*)$. Suppose $(\hat{x}, \hat{y}) \notin \psi(z^*)$. Then there is some $(x^*, y^*) \in \psi(z^*)$ and an $\varepsilon > 0$ such that $u(x^*, y^*) \geqslant u(\hat{x}, \hat{y}) + \varepsilon$.

Now, since u is an upper semicontinuous function, $\lim_{n \to \infty} \sup u(x^n, y^n) \leqslant u(\hat{x}, \hat{y})$. Thus, there is N_1 such that for $n \geqslant N_1$, $u(x^n, y^n) \leqslant u(\hat{x}, \hat{y}) + \varepsilon/3$. Consequently, for $n \geqslant N_1$,

$$u(x^*, y^*) \geqslant u(x^n, y^n) + 2\varepsilon/3. \tag{3}$$

Choose $0 < \lambda < 1$ such that $(1-\lambda)[u(0, 0) - u(x^*, y^*)] \geqslant -\varepsilon/3$. We claim that there is an N_2 such that for $n \geqslant N_2$, $(\lambda x^*, \lambda y^*) \in \phi(z^n)$. To see this, observe that $(0, 0) \in \mathfrak{T}$ and convexity of $\mathfrak{T}$ imply that $(\lambda x^*, \lambda y^*) \in \phi(\lambda z^*)$. Since $z^n \to z^*$, there is N_2 such that for $n > N_2$, $z^n \geqslant \lambda z^*$. Thus $\delta \lambda y^* - \lambda x^* \geqslant (\delta - 1)\, \lambda z^* \geqslant (\delta - 1)\, z^n$, establishing our claim.

Since $(x^n, y^n) \in \psi(z^n)$, for $n \geqslant N_2$,

$$\begin{aligned} u(x^n, y^n) \geqslant u(\lambda x^*, \lambda y^*) &\geqslant \lambda u(x^*, y^*) + (1-\lambda)\, u(0, 0) \\ &= u(x^*, y^*) + (1-\lambda)[u(0, 0) - u(x^*, y^*)] \\ &\geqslant u(x^*, y^*) - \varepsilon/3. \end{aligned}$$

Using this in (3) for $n \geqslant \text{Max}(N_1, N_2)$,

$$u(x^*, y^*) \geqslant u(x^n, y^n) + 2\varepsilon/3 \geqslant u(x^*, y^*) + \varepsilon/3,$$

which leads to a contradiction and completes the proof.

Proof of Theorem 4. Define Q: $S \to \mathfrak{P}(R^n_+)$, where for $z \in S$, $Q(z) = \{x \in R^n_+ : (x, y) \in \psi(z)\}$. We will show that this correspondence Q satisfies all the requirements of Kakutani's fixed-point theorem (Debreu [3, p. 26]).

Lemma 2 implies that Q is a non-empty, convex-valued correspondence. It also implies that Q is upper semicontinuous. To see this, take an arbitrary $z^* \in X$. Let $z^n \in S$, with $z^n \to z^*$ as $n \to \infty$. Let $x^n \in Q(z^n)$, and $x^n \to \hat{x}$ as $n \to \infty$. We have to show that $\hat{x} \in Q(z^*)$. Since $x^n \in Q(z^n)$, there is y^n such that $(x^n, y^n) \in \psi(z^n)$. This means $(x^n, y^n) \in \phi(z^n)$, and by compactness of S', we can pick a subsequence $(x^{n'}, y^{n'})$ tending to $(\hat{x}, \hat{y}) \in S'$. By the lemma, $(\hat{x}, \hat{y}) \in \psi(z^*)$ and the claim is proved.

Thus, all the conditions of Kakutani's fixed point theorem are fulfilled, and there exists $x^0 \in Q(x^0)$. This means there is some y^0 such that $(x^0, y^0) \in \psi(x^0)$, i.e.,

$$u(x^0, y^0) \geqslant u(x, y) \qquad \text{for all } (x, y) \in \phi(x^0).$$

But $(x^0, y^0) \in \phi(x^0)$ implies $x^0 \leqslant y^0$, and we obtain from A4 that $(x^0, x^0) \in \mathfrak{T}$, and $u(x^0, x^0) \geqslant u(x^0, y^0) \geqslant u(x, y)$ for all $(x, y) \in \mathfrak{T}$, with $\delta y - x \geqslant \delta x^0 - x^0$. Given δ-normality, there is $(x', y') \in \phi(x^0)$ such that $u(x', y') > u(0, 0)$. Thus $u(x^0, x^0) > u(0, 0)$, and hence x^0 is a discounted golden-rule stock. We can now state the principal result of this paper.

THEOREM 5. *If $\mathfrak{E}$ satisfies* A1–A5 *and is δ-normal, there exists a non-trivial stationary optimal stock.*

Proof. The proof is a simple consequence of Theorems 4 and 2.

Flynn [5] establishes a version of Theorem 5 under the additional assumption of δ-productivity. A natural question arises as to whether our generalization is non-vacuous, i.e., there exist economies satisfying the hypotheses of Theorem 5 (and Theorem 4), whose technologies are not δ-productive and for which there exists a non-trivial SOS. That this is indeed so can be seen by the following example.

EXAMPLE 2. Let $f(x) = 2x$ for $0 \leqslant x \leqslant 1$ and $f(x) = 2 + (x-1)/2$ for $x \geqslant 1$. Let $\mathfrak{T} = \{(x, y) \in R^n_+ : 0 \leqslant y \leqslant f(x)\}$, $u(x, y) = 2f(x) - y$ and $\delta = \frac{1}{2}$. $\mathfrak{E}$ satisfies A3 with $\beta = 3$ and it is easy to check that the economy $\mathfrak{E}$ also satisfies the remaining assumptions A1, A2, A4, and A5.

Now $(\bar{x}, \bar{y}) \equiv (1, 2) \in \mathfrak{T}$. Certainly $\delta \bar{y} - \bar{x} = 0$ and $u(\bar{x}, \bar{y}) = 2 > u(0, 0)$. Hence $\mathfrak{E}$ is δ-normal. Also, for any $(x, y) \in \mathfrak{T}$, $\delta y - x \leqslant \frac{1}{2} f(x) - x \leqslant 0$, since for $x \geqslant 1$, $f(x) \leqslant 2x$. Thus, there cannot exist any $(x, y) \in \mathfrak{T}$ such that $x \ll \delta y$ and $\mathfrak{T}$ is not δ-productive.

Next, we claim that $x^* = 1$ is a discounted golden-rule stock. Pick any $(x, y) \in \mathfrak{T}$ such that $x \leqslant (1-\delta)\ x^* + \delta y$. Then $y \geqslant 2x - 1$ and $u(x, y) \leqslant 2f(x) - 2x + 1$. Now

$$u(x, y) \leqslant 2(2x) - 2x + 1 \leqslant 3 \qquad \text{for} \quad 0 \leqslant x \leqslant 1$$

and

$$u(x, y) \leqslant 2(2 + \tfrac{1}{2}(x-1)) - 2x + 1 \leqslant 3 \qquad \text{for} \quad x \geqslant 1.$$

In either case, $u(x, y) \leqslant u(1, 1)$ and our claim is proved.

It should be noted that $x^* = 1$ is also a non-trivial SOS by Theorem 2.

Finally, we present an example of an economy which satisfies all the assumptions of Theorem 2 of McKenzie [9], but which has only a trivial SOS.

EXAMPLE 3. Let $\mathfrak{T} = \{(x, y) \in R_+^2 : 0 \leqslant y \leqslant 2x^{1/2}\}$, $\delta = \frac{1}{2}$, and $u(x, y) = x - 2y$. For $(\hat{x}, \hat{y}) = (\frac{1}{4}, 1) \in \mathfrak{T}$, we have $\delta\hat{y} \gg \hat{x}$. Note that our economy satisfies A1–A4 and I and II of McKenzie [9, pp. 196–198]. For any program $\{k\}_0^\infty$ with $0 < k \leqslant 4$, $\sum_{t=0}^{\infty} \delta^t u(k, k) < 0$ and is dominated by the program $\{k(t)\}_0^\infty$ with $k(0) = k$ and $k(t) = 0$, $t = 1, 2,\ldots$. Since there is no stationary program $\{k\}_0^\infty$ with $k > 4$, $\{0\}_0^\infty$ is the unique stationary optimal program.

Remark 2. It is worth pointing out that McKenzie defines a non-trivial SOS only for δ-productive technologies and as one which is a local turnpike. In this sense, (0) in Example 3 is non-trivial because the technology is δ-productive and (0) is a global turnpike.

REFERENCES

1. W. A. BROCK, On the existence of weakly maximal programmes in a multi-sector economy, *Rev. Econ. Stud.* **37** (1970), 275–280.
2. D. CASS AND K. SHELL, The structure and stability of competitive dynamical systems, *J. Econ. Theory* **12** (1976), 31–70.
3. G. DEBREU, "Theory of Value," Wiley, New York, 1959.
4. W. D. DECHERT AND K. NISHIMURA, A complete characterization of optimal growth paths in an aggregated model with a non-concave production function, *J. Econ. Theory* **31** (1983), 332–354.
5. J. FLYNN, The existence of optimal invariant stocks in a multi-sector economy, *Rev. Econ. Stud.* **47** (1980), 809–811.
6. T. HANSEN AND T. C. KOOPMANS, On the definition and computation of a capital stock invariant under optimization. *J. Econ. Theory* **5** (1972), 487–523.
7. O. L. MANGASARIAN, "Non-Linear Programming," McGraw-Hill, New York, 1969.
8. L. W. MCKENZIE, "Optimal Economic Growth and Turnpike Theorems," Discussion Paper No. 79-1, University of Rochester, 1979; also *in* "Handbook of Mathematical Economics" (K. J. Arrow and M. Intrilligator, Eds.), North-Holland, New York, in press.
9. L. W. MCKENZIE, A primal route to the turnpike and Lyapunov stability, *J. Econ. Theory* **27** (1982), 194–209.
10. B. PELEG, A weakly maximal golden-rule program for a multi-sector economy, *Int. Econ. Rev.* **14** (1973), 574–579.
11. B. PELEG AND H. E. RYDER, JR., On optimal consumption plans in a multi-sector economy, *Rev. Econ. Stud.* **39** (1972), 159–169.
12. B. PELEG AND H. E. RYDER, JR., The modified golden-rule of a multi-sector economy, *J. Math. Econ.* **1** (1974), 193–198.
13. W. R. S. SUTHERLAND, "Optimal Development Programs when the Future Utility is Discounted," Ph.D. dissertation, Brown University, 1967.
14. W. R. S. SUTHERLAND, On optimal development in a multi-sectoral economy: The discounted case, *Rev. Econ. Stud.* **37** (1970), 585–589.
15. M. L. WEITZMAN, Duality theory for infinite horizon convex models, *Manage. Sci.* **19** (1973), 783–789.